Integrable $(2k)$-Dimensional Hitchin Equations

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April 26, 2016

Abstract

This letter describes a completely-integrable system of Yang-Mills-Higgs equations which generalizes the Hitchin equations on a Riemann surface to arbitrary k-dimensional complex manifolds. The system arises as a dimensional reduction of a set of integrable Yang-Mills equations in $4k$ real dimensions. Our integrable system implies other generalizations such as the Simpson equations and the non-abelian Seiberg-Witten equations. Some simple solutions in the $k = 2$ case are described.

MSC: 81T13, 53C26.

Keywords: gauge theory, Higgs, integrable system.

1 Introduction

This note concerns completely-integrable systems of Yang-Mills-Higgs equations, and in particular those which may be viewed as higher-dimensional generalizations of the two-dimensional Hitchin equations (the self-duality equations on a Riemann surface). Let us begin by briefly setting out the notation.

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We denote local coordinates on $\mathbb{R}^n$ by x^μ with $\mu = 1, \ldots, n$. For simplicity we take the gauge group to be SU(2) throughout. A gauge potential A_μ takes values in the Lie algebra $\mathfrak{su}(2)$, so each of $A_1, \ldots, A_n$ is an anti-hermitian 2×2 matrix. The curvature (gauge field) is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. A Higgs field Φ takes values in the Lie algebra, or, if complex, in the complexified Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Its covariant derivative is $D_\mu = \partial_\mu \Phi + [A_\mu, \Phi]$, and gauge transformations act by $\Phi \mapsto \Lambda^{-1}\Phi\Lambda$.

The prototype system is the simplest 2-dimensional reduction [14] of the 4-dimensional anti-self-dual Yang-Mills equations

$$F_{12} + F_{34} = 0, \quad F_{13} + F_{42} = 0, \quad F_{14} + F_{23} = 0. \tag{1}$$

This reduction can be written as a conformally-invariant system on the complex plane $\mathbb{C}$, or more generally on a Riemann surface [12], and is effected as follows. If we take all the fields to depend only on the coordinates (x^1, x^2), and we define a complex coordinate $z = x^1 + ix^2$ and a complex Higgs field $\Phi = A_3 + iA_4$, then (1) reduces to the Hitchin equations

$$D_{\bar{z}}\Phi = 0, \quad F_{z\bar{z}} + \tfrac{1}{4}[\Phi, \Phi^*] = 0. \tag{2}$$

Several higher-dimensional generalizations of (2) have been introduced and studied over the years. But most such generalizations lack a notable property of the original system (2), namely its complete integrability. The purpose of this note is to describe some features, and some solutions, of an integrable $(2k)$-dimensional generalization of (2).

Let us focus specifically on generalizations to $2k$ real (or k complex) dimensions which involve $2k$ real (or k complex) Higgs fields. Such systems may naturally be viewed as dimensional reductions of pure-gauge systems in $4k$ dimensions, satisfying linear relations on curvature such as (1). Of greatest interest are those that have the eigenvalue form [4]

$$F_{\mu\nu} = \tfrac{1}{2} T_{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{3}$$

where $T_{\mu\nu\alpha\beta}$ is totally-skew, because the Bianchi identities then imply that the gauge field satisfies the second-order Yang-Mills equations.

Perhaps the best-known example is the 'octonionic' system of [4], which has $k = 2$. This may be written

$$\begin{aligned}
F_{12} + F_{34} + F_{56} + F_{78} &= 0, \\
F_{13} + F_{42} + F_{57} + F_{86} &= 0, \\
F_{14} + F_{23} + F_{76} + F_{85} &= 0, \\
F_{15} + F_{62} + F_{73} + F_{48} &= 0, \\
F_{16} + F_{25} + F_{38} + F_{47} &= 0, \\
F_{17} + F_{82} + F_{35} + F_{64} &= 0, \\
F_{18} + F_{27} + F_{63} + F_{54} &= 0.
\end{aligned} \tag{4}$$

Whereas the prototype (1) is essentially based on the quaternions, this system (4) is based on the octonions: the components of $T_{\mu\nu\alpha\beta}$ are constructed from the Cayley numbers. It is invariant under the group Spin(7), and its 7-dimensional reduction is invariant under G_2. We now reduce to four dimensions by requiring the fields to depend only on the variables (x^1, x^2, x^5, x^6), defining two complex variables and two complex Higgs fields by

$$z^1 = x^1 + ix^2, \quad z^2 = x^5 + ix^6, \quad \Phi_1 = A_5 + iA_6, \quad \Phi_2 = A_7 + iA_8. \tag{5}$$

Then the reduction of (4) is

$$\begin{aligned}
F_{1\bar{1}} + F_{2\bar{2}} + \tfrac{1}{4}[\Phi_1, \Phi_1^*] + \tfrac{1}{4}[\Phi_2, \Phi_2^*] &= 0, \\
F_{12} - \tfrac{1}{4}[\Phi_1, \Phi_2] &= 0, \\
D_{\bar{1}}\Phi_1 - D_2\Phi_2^* = 0, \ D_{\bar{2}}\Phi_1 + D_1\Phi_2^* &= 0.
\end{aligned} \tag{6}$$

Here the subscript 1 in $F_{1\bar{1}}$ and D_1 refers to z^1, whereas $\bar{1}$ refers to the complex conjugate variable $\bar{z}^1$. The equations (6) are more familiar in the $\mathbb{R}^4$ (real) form

$$\left(F - \tfrac{1}{2}[\Phi \wedge \Phi]\right)^+ = 0, \quad (D\Phi)^- = 0, \quad D * \Phi = 0, \tag{7}$$

where $\Phi = \Phi_\mu dx^\mu$ is a Lie-algebra-valued 1-form formed from the four real Higgs fields. The 'plus' superscript denotes the self-dual part of a 2-form, and the 'minus' superscript the anti-self-dual part. This system has appeared in several contexts over the years [6, 2, 13, 11, 8, 3], and has variously been referred to as the non-abelian Seiberg-Witten equations or the

Kapustin-Witten equations. Known solutions include several obtained using a generalized 't Hooft ansatz [7].

A different generalization of (2), defined on any Kähler manifold, is one attributed to Simpson [18]. In k complex dimensions, with complex coordinates z^a, $a = 1, \ldots, k$, it takes the form

$$F_{1\bar{1}} + \ldots + F_{k\bar{k}} + \tfrac{1}{4}[\Phi_1, \Phi_1^*] + \ldots + \tfrac{1}{4}[\Phi_k, \Phi_k^*] = 0,$$
$$F_{ab} = 0, \ [\Phi_a, \Phi_b] = 0, \ D_{\bar{a}}\Phi_b = 0. \tag{8}$$

Note that for $k = 1$, this system reduces to the prototype (2). For $k = 2$, it clearly it implies (6). The converse is not true in general, but it is if one imposes appropriate global conditions: in particular for smooth fields on a compact Kähler surface, it has recently been shown that (8) and (6) are equivalent [19].

2 An integrable version

Another approach to generalizing the basic 4-dimensional system (1) is to look for higher-dimensional versions which are completely-integrable [20]. For simplicity, we begin with the case $k = 2$. An integrable 8-dimensional Yang-Mills system is

$$\begin{aligned}
F_{12} + F_{34} = F_{56} + F_{78} &= 0, \\
F_{13} + F_{42} = F_{57} + F_{86} &= 0, \\
F_{14} + F_{23} = F_{76} + F_{85} &= 0, \\
F_{15} = F_{26} = F_{37} &= F_{48}, \\
F_{16} = F_{52} = F_{83} &= F_{47}, \\
F_{17} = F_{28} = F_{53} &= F_{64}, \\
F_{18} = F_{72} = F_{36} &= F_{54},
\end{aligned} \tag{9}$$

which clearly implies the octonionic equations (4). The system (9) has the symmetry group $[\mathrm{Sp}(1) \times \mathrm{Sp}(2)] / \mathbb{Z}_2 \subset \mathrm{SO}(8)$, which corresponds to a quaternionic Kähler structure [17]. The ADHM construction of instantons [1] generalizes to this case [17, 5, 15]. Consider now the reduction to four dimensions, with the same complex variables (5) as before. Then (9) reduces to

$$D_{\bar{a}}\Phi_b = 0,\ F_{a\bar{b}} + \tfrac{1}{4}[\Phi_a, \Phi_b^*] = 0,\ [\Phi_a, \Phi_b] = 0,\ F_{ab} = 0,\ D_{[a}\Phi_{b]} = 0, \quad (10)$$

where $a, b \in \{1, 2\}$. This system is even more overdetermined than (8). So we have a string of implications, where (10) implies (8) implies (6) implies the four-dimensional Yang-Mills-Higgs equations (the reduction of pure Yang-Mills from eight dimensions).

Generalizing (10) to k complex dimensions is straightforward: we simply allow the indices a, b to range from 1 to k. The system (10) has a very large symmetry group, since it involves only the holomorphic structure of the underlying complex manifold. This becomes clearer if we define

$$\Phi = \sum_a \Phi_a \, dz^a$$

as a $(1, 0)$-form with values in the complexified Lie algebra: then (10) can be written

$$D\Phi = 0,\ F^{1,1} + \tfrac{1}{4}[\Phi \wedge \Phi^*] = 0,\ [\Phi \wedge \Phi] = 0,\ F^{2,0} = 0, \quad (11)$$

where D now denotes the covariant exterior derivative. By contrast, the less-overdetermined systems (8) and (6) depend on an underlying geometric structure, and have less symmetry.

The system (10) is completely-integrable by virtue of being the consistency condition for a 'Lax $(2k)$-tet', namely

$$\eth_a = D_a + \tfrac{1}{2}\zeta\Phi_a, \quad \eth_{\bar{a}} = D_{\bar{a}} + \tfrac{1}{2}\zeta^{-1}\Phi_a^*, \quad (12)$$

where ζ is a complex parameter. The integrability conditions

$$[\eth_a, \eth_b] = 0 = [\eth_a, \eth_{\bar{b}}]$$

for all ζ are equivalent to the equations (10).

3 Some solutions

The aim now is to describe some solutions of (10); these will therefore also be solutions of the other systems (8), and (7) in the $k = 2$ case. The equations (10) or (11) are defined on any k-dimensional complex manifold, and in general one may also allow singularities. For example, in the $k = 1$ case on a compact Riemann surface of genus g, smooth solutions of (2) exist only when $g \geq 2$; on the 2-sphere and the 2-torus, solutions necessarily have singularities [12]. Note that the functions $G_{ab} = \mathrm{tr}(\Phi_a \Phi_b)$ are holomorphic, by virtue of the equations (11). In what follows, we look for solutions which are smooth on $\mathbb{C}^2$, and for which G_{ab} is a polynomial in z^a. So they may also be viewed as being defined on the projective plane $\mathbb{CP}^2$, with a singularity on the line at infinity.

To illustrate, let us first consider the abelian case, with the fields being diagonal, namely $\Phi_a = \phi_a \sigma_3$, where $\sigma_3 = \mathrm{diag}(1, -1)$. Then the equations (11) are easily solved. The gauge field vanishes, and therefore we may take the gauge potential to vanish as well. The remaining equations give $\Phi = d\theta$, where $\theta(z^a)$ is an arbitrary polynomial on $\mathbb{C}^2$. This is the general abelian solution.

For the non-abelian SU(2) case, we adopt a simplifying ansatz which is familiar from the lower-dimensional version [10]. Namely let us assume that the gauge potential is diagonal: in other words, $A_{\bar{a}} = h_{\bar{a}} \sigma_3$. (It should be emphasized that there are solutions for which this assumption does not hold.) Then the general local solution is determined by a holomorphic function $\theta(z^a)$, plus a solution $u = u(\theta, \bar{\theta})$ of the elliptic sinh-Gordon equation

$$\partial_\theta \partial_{\bar{\theta}} \log |u| = \tfrac{1}{4} \left(|u|^2 - |u|^{-2} \right). \tag{13}$$

In terms of these, the Higgs fields are given by

$$\Phi_a = (\partial_a \theta) \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix},$$

and the functions determining the gauge potential are

$$h_{\bar{a}} = -\tfrac{1}{2} \partial_{\bar{a}} \log(u).$$

Note that one solution of (13) is $u = 1$, but this is effectively the abelian case of the previous paragraph. In order to get genuine non-abelian fields, we choose $\theta(z^a)$ to have branch singularities, and then to get smooth fields one needs $u \neq 1$. The simplest such fields are embeddings of solutions of (2) on $\mathbb{C}$ into $\mathbb{C}^2$, depending on z^a only via a fixed linear combination $z = \alpha z^1 + \beta z^2$.

For example, $\theta(z) = z^{3/2}$ gives an embedding of the 'one-lump' solution on $\mathbb{C}$ [21]. Some simple solutions that are not of this embedded type are as follows.

Let $P(z^a)$ be a polynomial of degree at least two, and take $\theta = \frac{2}{3}P^{3/2}$. This gives Higgs fields of the form

$$\Phi_a = (\partial_a P) \begin{pmatrix} 0 & P\mathrm{e}^{\psi/2} \\ \mathrm{e}^{-\psi/2} & 0 \end{pmatrix}, \tag{14}$$

where $\psi(P, \bar{P})$ satisfies

$$\partial_P \partial_{\bar{P}} \psi = \tfrac{1}{2}\left(|P|^2 \mathrm{e}^{\psi} - \mathrm{e}^{-\psi}\right). \tag{15}$$

We now need a smooth solution of (15) satisfying the boundary condition $\psi \sim -\log|P|$ as $|P| \to \infty$. There exists a unique such solution, which is essentially a Painlevé-III function [9, 21]. In fact, if we define $h(t) = t^{-1/3}\mathrm{e}^{-\psi/2}$, where $t = |P|^{3/2}$, then (15) becomes an equation of Painlevé-III type, namely

$$h'' - \frac{(h')^2}{h} + \frac{h'}{t} + \frac{4}{9h} - \frac{4h^3}{9} = 0. \tag{16}$$

This has a unique solution with the required asymptotics.

The upshot is that any polynomial $P(z^a)$ gives a solution of (11) which is smooth on $\mathbb{C}^2$ and has

$$G_{ab} = \mathrm{tr}\,(\Phi_a \Phi_b) = 2P(\partial_a P)(\partial_b P).$$

It appears (see for example the figure below) that the gauge field $F_{\mu\nu}$ is concentrated around the zero-set of P. In the general k-complex-dimensional case, one expects the gauge field to be concentrated around a submanifold of complex codimension 1, and for the field to be approximately abelian elsewhere.

The simplest case has P quadratic, so that $P(z^a) = 0$ is a conic. Figure 1 is a plot of the norm $|F|$ of the gauge field, on the real slice $(z^1, z^2) \in \mathbb{R}^2$, for the solutions corresponding to the choices $P(z^a) = 2(z^1)^2 + (z^2)^2 - 4$ (on the left), and $P(z^a) = z^1(z^1 + 2z^2)$ (on the right). Here $|F|$ is computed using the metric $ds^2 = dz^1\, d\bar{z}^1 + dz^2\, d\bar{z}^2$ on $\mathbb{C}^2$, which leads to the formula

$$|F| = \left|\mathrm{e}^{-\psi} - |P|^2\mathrm{e}^{\psi}\right| \left(|\partial_1 P|^2 + |\partial_2 P|^2\right). \tag{17}$$

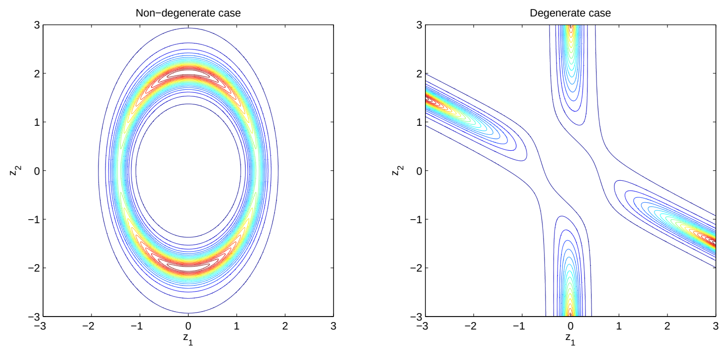

Figure 1: Contour plots of the gauge field $|F(z^a)|$ for $z^a \in \mathbb{R}^2$, with $P(z^a) = 2(z^1)^2 + (z^2)^2 - 4$ and $P(z^a) = z^1(z^1 + 2z^2)$ respectively.

The figures were generated by solving (16) numerically to get ψ, and then using this formula (17). Clearly $|F|$ is concentrated around the conic $P(z^a) = 0$. The right-hand case corresponds to a degenerate conic, and is the reduced version of what was called 'instantons at angles' [16] for solutions of (9).

4 Remarks

There are some compact complex manifolds X on which smooth solutions of (11) exist. As a trivial example, one could take X to be a product $S \times X'$, where S is a Riemann surface of genus at least two, and X' is any other manifold: then a solution of (2) on S is also a solution of (11) on $S \times X'$. The moduli space of solutions on any compact manifold, if it is non-empty, has a natural L^2 metric, which on general grounds one expects to be hyperkähler. Even more generally, one could allow singularities of a specified type, or equivalently for the ambient space to be non-compact. In this latter case, some of the parameters in the solution space may have L^2 variation, giving rise to a moduli space with a well-defined metric. Analysing the possible moduli space geometries which arise in this way would be worthwhile, although a considerable task.

In this note, we have focused on a particular type of reduction of the integrable system (9), and of its $(4k)$-dimensional generalization. There are several other dimensional reductions of the octonionic system (4) which are of interest: see, for example, reference [3]. In each case, the appropriate reduction of (9) gives an integrable sub-system, and hence a source of solutions.

Acknowledgments. This work was prompted by a communication from Sergey Cherkis. The author acknowledges support from the UK Particle Science and Technology Facilities Council, through the Consolidated Grant ST/L000407/1.

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